Department of Mathematics

SEM - 6

Course - BMH6DSE33

Group Theory - II

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Them

Notes by Rima Dutta

Fundamental theorem on finite Abelian Groups :-

A finite abelian group is isomorphic to a direct product of cyclic groups of prime-power order, where the ~~der~~ decomposition is unique up to the order in which the factors are written.

Ex. 1. Let $o(G) = 28 = 7 \times 2^2$

$$2 = 2$$
$$= 1 + 1.$$

Then G is isomorphic to $\mathbb{Z}_{2^2} \times \mathbb{Z}_7$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$.

2. Let $o(G) = 108 = 2^2 \times 3^3$.

$$2 = 2 \qquad 3 = 3$$
$$= 1+1 \qquad = 2+1$$
$$= 1+1+1$$

$\therefore$ G is isomorphic to $\mathbb{Z}_{2^2} \times \mathbb{Z}_{3^3}$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_3$,
$\mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_{2^1} \times \mathbb{Z}_2 \times \mathbb{Z}_{3^3}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_3$,
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

i.e. G is isomorphic to $\mathbb{Z}_{108}$, $\mathbb{Z}_{36} \times \mathbb{Z}_3$, $\mathbb{Z}_{12} \times \mathbb{Z}_3 \times \mathbb{Z}_3$,
$\mathbb{Z}_{54} \times \mathbb{Z}_2$, ~~$\mathbb{Z}_{18} \times \mathbb{Z}_2$~~ $\mathbb{Z}_{18} \times \mathbb{Z}_6$, $\mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_3$. (অন্যভাবে পাইলে তম order -এ group গুলোর নির্ণয় সেইভাবে [illegible] আছে)

Characteristics Subgroups.

Definition :- A subgroup H of a group G is said to be characteristics subgroup of G, if for every automorphism ϕ of G, $\phi(H) \leq H$ holds, i.e. if every automorphism of the mother group maps the subgroup to ~~~~ within itself. It is denoted by ~~ie~~. H char G. and written by H is characteristics in G.

i.e. Given H char G, every automorphism of G induces an automorphism of the quotient group G/H, which yeilds an map $Aut(G) \to Aut(G/H)$.

Note :- 1. If G has ~~~~ the only one subgroup H of a finite index, then H is characteristics in G.

2. Every subgroup of a cyclic group is characteristic in the mother group.

Simple Group

Definition :- A group $G (\neq \{e\})$ is called simple if G has no non trivial normal subgroup [i.e. G and $\{e\}$ are the only normal subgroups of G].

Exercise : Prove that A_4 is not a simple group.

Solution : Now, $A_4 = \{e, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$.

Let $K = \{e, (12)(34), (13)(24), (14)(23)\}$.

Then K is a subgroup of A_4 and K is the only subgroup of order 4.

Hence, K is a normal subgroup of A_4.

Thus, A_4 is not simple group.

Theorem :- Let H be a normal subgroup of A_n ($n \geq 5$). If H contains 3-cycle then H contains all 3-cycles.

Proof :- Let $(abc) \in H$.

Let $(u\,v\,w) \in A_n$.

then $\exists$ an element $\alpha \in S_n$ such that $\alpha(a) = u$, $\alpha(b) = v$, $\alpha(c) = w$.

Also, $\alpha(abc)\alpha^{-1} = (u\,v\,w)$.

If $\alpha \in A_n$ then $(u\ v\ w) = \alpha(a\ b\ c)\alpha^{-1} \in \alpha H \alpha^{-1} \subseteq H$

since H is normal subgroup of A_n.

$\therefore$ $(u\ v\ w) \in H$.

Let $\alpha \notin A_n$

Then α is an odd permutation.

Since $n \geq 5$, so there exists $x, y \in I_n$ such that $x, y \notin \{a, b, c\}$.

Now, $(x\ y) \in S_n$ is an odd permutation and hence $\alpha(x\ y) \in A_n$ and therefore,

$(\alpha(x\ y))(a\ b\ c)(\alpha(x\ y))^{-1} \in H$. [Since H is normal in A_n]

ie $\alpha(x\ y)(a\ b\ c)(x\ y)\alpha^{-1} \in H$

ie. $\alpha(a\ b\ c)(x\ y)(x\ y)\alpha^{-1} \in H$ [Since any two disjoint cycles commute to each other]

ie. $\alpha(a\ b\ c)\alpha^{-1} \in H$

ie. $(u\ v\ w) \in H$.

Hence, H contains all 3-cycles.

Theorem :- Let H be a normal subgroup of $A_n (n \geq 5)$. If H contains a 3-cycle then $H = A_n$.

Proof :- Since H contains a 3-cycle so H contains all 3-cycles.

Let $\alpha \in A_n$.

then α is an even permutation.

Hence there exists even number of 2-cycles $\alpha_1, \alpha_2, \dots, \alpha_{2K}$ such that $\alpha = \alpha_1 \alpha_2 \dots \alpha_{2K}$

Let $\alpha_1 = (a\ b)$, $\alpha_2 = (c\ d)$.

then $\alpha_1 \alpha_2 = (a\ b)(c\ d) = (a\ c\ b)(a\ c\ d)$.

Since $2K$ is even, proceeding as above, we find that α can be expressed as a ~~prou~~ product of 3-cycles.

Hence $\alpha \in H$ i.e. $A_n \subseteq H$.

Thus, $H = A_n$.

Theorem :- Let H be a normal subgroup of $A_n (n \geq 5)$. If H contains a product of two disjoint 2-cycles then $H = A_n$.

Proof:- Let $(a\ b)$ and $(c\ d)$ be two disjoint 2-cycles such that $(a\ b)(c\ d) \in H$.

Since $n \geq 5$, so there exists, ~~u~~ $u \in I_n$ such that

$u \notin \{a, b, c, d\}$.

Now, $(a\,b\,u) \in A_n$.

Hence $(a\,b\,u)(a\,b)(c\,d)(a\,b\,u)^{-1} \in H$ [$\because H$ is normal in A_n]

ie. $(a\,b\,u)(a\,b)(c\,d)(a\,u\,b) \in H$

~~ie.~~ ie. $(u\,b)(c\,d) \in H$.

Hence $(a\,b)(c\,d)(u\,b)(c\,d) \in H$.

ie. $(a\,b)(u\,b) \in H$.

ie. $(a\,b\,u) \in H$.

Thus, H contains all 3-cycles and hence $H = A_n$.

Exercise :- Prove that A_5 is a simple group.

Solution : Let H be a normal subgroup of A_5 such that $H \neq \{e\}$.

Since $H \neq \{e\}$, so there exists an element $\alpha \in H$ such $\alpha \neq e$.

Since $\alpha \in A_5$ and $\alpha \neq e$, then α is either a 3-cycle or a product of two disjoint 2-cycles or a 5-cycle.

Now, if α is a product of two disjoint 2-cycles then H contains a 3-cycle and hence $H = A_5$.

If α is a 3-cycle, then H contains all 3-cycles and hence $H = A_5$.

Let ~~H~~ α be a 5-cycle.

Let $\alpha = (a\ b\ c\ d\ f)$

Now, $(a\ b\ c) \in A_5$.

Since H is a normal subgroup of A_5, therefore,

$(a\ b\ c)(a\ b\ c\ d\ f)(a\ b\ c)^{-1} \in H$

i.e. $(a\ b\ c)(a\ b\ c\ d\ f)(a\ c\ b) \in H$

i.e. $(a\ d\ f\ b\ c) \in H$.

Hence $(a\ b\ c\ d\ f)^{-1} \cdot (a\ d\ f\ b\ c) \in H$

i.e. $(a\ f\ d\ c\ b)(a\ d\ f\ b\ c) \in H$

i.e. $(a\ c\ f) \in H$.

$\therefore$ H contains all 3-cycles and $H = A_5$.

Hence, A_5 is a simple group.